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Natural Choice of L_1-Approximants

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1. INTRODUCTION

Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space. For $1 \leqslant s < \infty$ denote by $L_s(\Omega, \mathscr{A}, \mu)$ the system of all equivalence classes of $\mathscr{A}$-measurable real functions $f: \Omega \to \mathbb{R}$ with $\|f\|_s := [\int |f|^s \, d\mu]^{1/s} < \infty$.

For $\varnothing \neq C \subset L_1$ and $f \in L_1$ let $\mu_1(f \mid C)$ be the set of all best $\| \;\|_1$-approximants of f in C, i.e., the set of all $g \in C$ with

$$\|f - g\|_1 = \inf\{\|f - h\|_1 : h \in C\}.$$

It is known that even for nice C's best $\| \;\|_1$-approximants of f in C may not exist, e.g., it may happen that C is a $\| \;\|_1$-closed linear subspace and $\mu_1(f \mid C) = \varnothing$ for al $f \notin C$ (see [11, p. 100]). However, for many important C's best $\| \;\|_1$-approximants always exist, e.g., for $\| \;\|_1$-closed convex lattices C (see [7]) or for finite dimensional subspaces $C \subset L_1$. But in all these cases best $\| \;\|_1$-approximants are rarely uniquely determined. Assume in the following that C is a $\| \;\|_1$-closed convex set and $\mu_1(f \mid C) \neq \varnothing$. Many investigations on L_1-approximation are concerned with the problem of characterizing "uniqueness" classes C, i.e., characterizing those C's allowing unique best $\| \;\|_1$-approximants (see Chap. I, Sect. 3 of [11]). We believe that searching for uniqueness classes could become less important because it turns out that in the class $\mu_1(f \mid C)$ of all best $\| \;\|_1$-approximants exactly one element is highly priviledged; it is among all best $\| \;\|_1$-approximants of f in C

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the best $\| \|_s$-approximant for all s near 1. More precisely there exists $m_1 \in \mu_1(f \mid C)$ such that for each other $g \in \mu_1(f \mid C)$ we have

$$\|f - m_1\|_s < \|f - g\|_s \qquad \text{for all sufficiently small } s > 1.$$

This best $\| \|_1$-approximant m_1 seems to be a natural and reasonable choice of a best $\| \|_1$-approximant of f in C. Moreover $m_1(f \mid C)$ has another prominent property: it is the $\| \|_1$-limit of the uniquely determined best $\| \|_s$-approximants of f in C for $s \downarrow 1$. From this convergence property it follows that the map $f \to m_1(f \mid C)$ has some nice algebraic properties.

The concept presented here contains the following cases: If C is the set of all constant functions and μ is a probability measure then $\mu_1(f \mid C)$ is the set of all medians of f, and the natural best $\| \|_1$-approximant $m_1 \in \mu_1(f \mid C)$ is the natural median of f which was introduced in [9]. For this special case it was shown by a direct calculation in [9] that the best $\| \|_s$-approximants of f in the system C of all constant functions converge to a median.

If $\mathscr{A}_0 \subset \mathscr{A}$ is a σfield and C is the system of all $\mathscr{A}_0$-measurable functions in L_1 then $\mu_1(f \mid C)$ is the set of all conditional medians of f given $\mathscr{A}_0$ (see [10, 12]), and the natural best $\| \|_1$-approximant $m_1 \in \mu_1(f \mid C)$ could be termed a natural conditional median of f given $\mathscr{A}_0$.

The presented concept of natural best $\| \|_1$-approximants can furthermore be applied to all $\| \|_1$-closed lattices $C \subset L_1$ fulfilling $aC + b \subset C$ for $a \geqslant 0$, $b \in \mathbb{R}$. These C's are exactly the systems considered in the theory of isotonic regression and approximation (see [2, 3, 4, 5, 8]): these systems allow the treatment of statistical problems under order restrictions.

2. The Results

Now we formalize the concept of "natural" best $\| \|_1$-approximants described in the Introduction.

1. Definition. Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space. Let $f \in L_1$ and $C \subset L_1$ be a $\| \|_1$-closed convex set. An element $m_1(f \mid C) \in \mu_1(f \mid C)$ is called a natural best $\| \|_1$-approximant of f in C if for each $g \in \mu_1(f \mid C)$, $g \neq m_1(f \mid C)$, there exists $s(g) > 1$ such that

$$\|f - m_1(f \mid C)\|_s < \|f - g\|_s \qquad \text{for all} \quad 1 < s \leqslant s(g). \qquad (*)$$

Obviously there exists at most one natural best $\| \|_1$-approximant of f in C. As, however, condition $(*)$ is a strong additional approximation property for a best $\| \|_1$-approximant, it seems doubtful whether a natural best $\| \|_1$-approximant exists in non-trivial cases. Condition $(*)$ can—except for the case of unique best $\| \|_1$-approximants—never be fulfilled if $\|f - g\|_s = \infty$ for $s > 1$, $g \in \mu_1(f \mid C)$. Therefore we will assume in the following that

$$f \in L_{1+} = \bigcup_{s>1} L_s \qquad \text{and} \qquad \varnothing \neq \mu_1(f \mid C) \subset L_{1+}.$$

Theorem 2 shows that these assumptions alone guarantee the existence of a natural best $\| \|_1$-approximant of f in C.

If $s > 1$ and $C \subset L_s$ is a $\| \|_s$-closed convex set it is well known, that for each $f \in L_s$ there exists a unique best $\| \|_s$-approximant of f in C; denote it by $\mu_s(f \mid C)$.

2. THEOREM. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and $C \subset L_1(\Omega, \mathscr{A}, \mu)$ a $\| \|_1$-closed convex set. Then for each $f \in L_{1+}$ with $\varnothing \neq \mu_1(f \mid C) \subset L_{1+}$ we have*

(i) *there exists a natural best $\| \|_1$-approximant of f in C, say $m_1(f \mid C)$,*

(ii) *$m_1(f \mid C)$ is the unique best $\| \|_1$-approximant of f in C minimizing $\int |f - g| \ln |f - g| \, d\mu$ among all best $\| \|_1$-approximants g of f in C,*

(iii) *$\mu_s(f \mid C \cap L_s)$ converges with $s \downarrow 1$ strongly in L_1 to $m_1(f \mid C)$.*

Proof. Let $D := \mu_1(f \mid C)$ be the set of all best $\| \|_1$-approximants of f in C. Since $f \in L_{1+}$, $D \subset L_{1+}$ and $\mu \mid \mathscr{A}$ is a finite measure, for each $g \in D$ there exists $s(g) > 1$ such that $f, g \in L_s$ for all $1 \leqslant s \leqslant s(g)$. Hence

$$\varphi_g(s) := \int |f - g|^s \, d\mu \in \mathbb{R} \qquad \text{for} \quad 1 \leqslant s \leqslant s(g), \; g \in D. \tag{1}$$

Since D is the set of best $\| \|_1$-approximants of f in C we have

$$\varphi_g(1) = \varphi_h(1) \qquad \text{for all} \quad g, h \in D. \tag{2}$$

We prove that there exists $g_0 \in D$ with

$$\varphi'_{g_0}(1) < \varphi'_g(1) \qquad \text{for each} \quad g \in D, \; g \neq g_0, \tag{3}$$

where $\varphi'_g(1) = (d/ds)\, \varphi_g(s) |_{s=1}$. Then (2) and (3) imply for each $g \in D$ with $g \neq g_0$ that

$$\int |f - g_0|^s \, d\mu = \varphi_{g_0}(s) < \varphi_g(s) = \int |f - g|^s \, d\mu$$

for sufficiently small $s > 1$; i.e., g_0 is a natural best $\| \|_1$-approximant of f in C. Thus to prove (i) it remains to prove (3). To this aim we give at first an explicit expression for $\varphi'_g(1)$, $g \in D$. Since $(d/ds)\, |f - g|^s = |f - g|^s$

$\ln|f-g| \geqslant -1/e$ for all $s \geqslant 1$ we obtain from (1) and the finiteness of $\mu \mid \mathscr{A}$ that $\sup_{1 \leqslant s \leqslant s_1} |(d/ds)\,|f-g|^s| \in L_1$ if $s_1 < s(g)$. Hence we can interchange integration and differentiation according to the Lebesgue theorem and obtain

$$\varphi_g'(1) = \int |f-g| \ln |f-g| \, d\mu \in \mathbb{R}, \qquad g \in D. \tag{4}$$

Let $\Phi(x) = x \ln x$ for $x > 0$ and $\Phi(0) = 0$. Denote by M the set of all $h \in D$ such that

$$\int \Phi(|f-h|)\, d\mu = \inf_{g \in D} \int \Phi(|f-g|)\, d\mu =: \alpha \in \mathbb{R}.$$

To prove (3) and hence (i) and (ii), it therefore remains to show according to (4) that M contains exactly one element.

At first we show that $M \neq \varnothing$. Let $g_n \in D$ with $\int \Phi(|f-g_n|)\, d\mu \searrow_{n \in \mathbb{N}} \alpha$ be given. Since $\Phi(x)/x \to_{x \to \infty} \infty$ and Φ is bounded from below, we obtain that $|f-g_n|$, $n \in \mathbb{N}$, and hence g_n, $n \in \mathbb{N}$, is uniformly integrable. Therefore there exists a $g_0 \in L_1$ and a subsequence g_n, $n \in \mathbb{N}_1$, with $g_n \to_{n \in \mathbb{N}_1} g_0$ weakly in L_1. Since D is a convex and $\| \ \|_1$-closed set, $D \subset L_1$ is weakly closed and hence $g_0 \in D$. Since $f - g_n \to_{n \in \mathbb{N}_1} f - g_0$ weakly and $\int |f-g_n|\, d\mu = \int |f-g_0|\, d\mu$, we obtain from Lemma 8 that

$$|f-g_n| \xrightarrow[n \in \mathbb{N}_1]{} |f-g_0| \qquad \text{weakly.} \tag{5}$$

Since Φ is convex and continuous on $I = [0, \infty)$, we obtain from (5) according to Lemma 6

$$\int \Phi(|f-g_0|)\, d\mu \leqslant \varliminf_{n \in \mathbb{N}_1} \int \Phi(|f-g_n|)\, d\mu = \alpha. \tag{6}$$

As $g_0 \in D$ we obtain from (6) that $M \neq \varnothing$. To prove (i) and (ii) therefore it remains to show

$$g_1, g_2 \in M \qquad \text{implies} \qquad g_1 = g_2. \tag{7}$$

We show at first that

$$\mu\{g_1 < f < g_2\} = 0, \qquad \mu\{g_2 < f < g_1\} = 0. \tag{8}$$

Let $B = \{g_1 < f < g_2\}$. Then we have

$$|f - \tfrac{1}{2}(g_1 + g_2)| \leqslant \tfrac{1}{2}\,|f-g_1| + \tfrac{1}{2}\,|f-g_2|,$$

where "$<$" holds on B. Hence $\mu(B) > 0$ implies

$$\int |f - \tfrac{1}{2}(g_1 + g_2)|\, d\mu < \tfrac{1}{2} \int |f - g_1|\, d\mu + \tfrac{1}{2} \int |f - g_2|\, d\mu. \tag{9}$$

Since $g_1, g_2 \in D = \mu_1(f \mid C) \subset C$ and $\frac{1}{2}(g_1 + g_2) \in C$ by the convexity of C, (9) yields a contradiction. Therefore $\mu\{g_1 < f < g_2\} = \mu(B) = 0$; by symmetry $\mu\{g_2 < f < g_1\} = 0$. Hence (8) holds. As $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$ is strictly convex we have

$$\Phi(|a - \tfrac{1}{2}(b_1 + b_2)|) \leqslant \tfrac{1}{2}\Phi(|a - b_1|) + \tfrac{1}{2}\Phi(|a - b_2|) \tag{10}$$

if $b_1, b_2 \leqslant a$ or $b_1, b_2 \geqslant a$, where "$<$" holds if additionally $b_1 \neq b_2$. Using (8) we may apply for μ-a.a. $\omega \in \Omega$ relation (10) to $a = f(\omega)$, $b_1 = g_1(\omega)$ and $b_2 = g_2(\omega)$. Therefore we obtain μ-a.e.

$$\Phi(|f - \tfrac{1}{2}(g_1 + g_2)|) \leqslant \tfrac{1}{2}\Phi(|f - g_1|) + \tfrac{1}{2}\Phi(|f - g_2|), \tag{11}$$

where "$<$" holds on the set $\{g_1 \neq g_2\}$. If $\mu\{g_2 \neq g_2\} > 0$, integration of (11) yields, as $\Phi(|f - g_i|) \in L_1$ by (4), that

$$\int \Phi(|f - \tfrac{1}{2}(g_1 + g_2)|)\, d\mu < \tfrac{1}{2} \int \Phi(|f - g_1|)\, d\mu + \tfrac{1}{2} \int \Phi(|f - g_2|)\, d\mu. \tag{12}$$

As $g_1, g_2 \in M \subset D$ and $D = \mu_1(f \mid C)$ is convex, we obtain $\frac{1}{2}(g_1 + g_2) \in D$. Since $g_1, g_2 \in M$, (12) yields a contradiction. Hence $g_1 = g_2$ μ-a.e., i.e., (7) is shown. Hence (i) and (ii) are proven.

It remains to prove (iii). As C is convex and $\|\ \|_1$-closed, $\varnothing \neq \mu_1(f \mid C) \subset C$ and as $\mu_1(f \mid C) \subset L_{1+}$ we obtain for s near by 1 that $\varnothing \neq C \cap L_s$ is convex and $\|\ \|_s$-closed. As $f \in L_s$ for s near by 1, the best $\|\ \|_s$-approximant $\mu_s(f \mid C \cap L_s)$ of f given $C \cap L_s$ exists and is uniquely determined. Let $s_n \downarrow 1$ and put $g_n := \mu_{s_n}(f \mid C \cap L_{s_n})$ and $m_1 := m_1(f \mid C)$. We prove $g_n \to_{n \in \mathbb{N}} m_1$ strongly with the help of the following three steps.

$$\overline{\lim_{n \in \mathbb{N}}} \int \Phi(|f - g_n|)\, d\mu \leqslant \int \Phi(|f - m_1|)\, d\mu, \tag{13}$$

$$g_n \xrightarrow[n \in \mathbb{N}_1]{} g_0 \text{ weakly implies } g_0 \in \mu_1(f \mid C)$$

$$\text{and } \int |f - g_n|\, d\mu \xrightarrow[n \in \mathbb{N}_1]{} \int |f - g_0|\, d\mu, \tag{14}$$

$$g_n \xrightarrow[n \in \mathbb{N}]{} m_1 \text{ weakly.} \tag{15}$$

Assume that (13)–(15) are proven. Then (15), (14) and Lemma 8 imply

$|f-g_n| \to_{n\in\mathbb{N}} |f-m_1|$ weakly. Hence (13) implies $|f-g_n| \to_{n\in\mathbb{N}} |f-m_1|$ strongly according to Lemma 7, whence (15) and Lemma 9 imply $m_1 - g_n = f - g_n - (f-m_1) \to_{n\in\mathbb{N}} 0$ strongly, i.e., (iii).

To (13): By the mean value theorem

$$\Phi(x) = x \ln x \leqslant \frac{x^s - x}{s-1} \qquad \text{for} \quad x \geqslant 0,\ s > 1$$

and hence

$$\Phi(|f-g_n|) \leqslant \frac{|f-g_n|^{s_n} - |f-g_n|}{s_n - 1}. \tag{16}$$

Using that $g_n = \mu_{s_n}(f \mid C \cap L_{s_n}) \in C$ and $m_1 \in \mu_1(f \mid C)$, we have

$$\int |f-g_n|^{s_n}\, d\mu \leqslant \int |f-m_1|^{s_n}\, d\mu$$

and

$$\int |f-m_1|\, d\mu \leqslant \int |f-g_n|\, d\mu.$$

Then integration of (16) yields

$$\int \Phi(|f-g_n|)\, d\mu \leqslant \frac{1}{s_n-1} \int (|f-m_1|^{s_n} - |f-m_1|)\, d\mu. \tag{17}$$

With $n \to \infty$, we obtain from (17) and (4) relation (13).

To (14): Since $g_n \in C$ and C is weakly closed we have $g_0 \in C$.

Let $h \in \mu_1(f \mid C) \subset L_{1+}$ be given. As $f - g_n \to_{n\in\mathbb{N}_1} f - g_0$ weakly we have

$$\begin{aligned} \|f-g_0\|_1 &\leqslant \varliminf_{n\in\mathbb{N}_1} \|f-g_n\|_1 \leqslant \varlimsup_{n\in\mathbb{N}_1} \|f-g_n\|_1 \leqslant \varlimsup_{n\in\mathbb{N}_1} \|f-g_n\|_{s_n} \\ &\leqslant \varlimsup_{n\in\mathbb{N}_1} \|f-h\|_{s_n} = \|f-h\|_1. \end{aligned} \tag{18}$$

As $g_0 \in C$ this implies $g_0 \in \mu_1(f \mid C)$. Therefore we may choose $h = g_0$ in (18) and obtain $\int |f-g_n|\, d\mu \to_{n\in\mathbb{N}_1} \int |f-g_0|\, d\mu$.

To (15): According to (13), $|f-g_n|$, $n \in \mathbb{N}$, and hence g_n, $n \in \mathbb{N}$, is uniformly integrable. Hence to each subsequence $\mathbb{N}_1 \subset \mathbb{N}$ there exist a subsequence $\mathbb{N}_2 \subset \mathbb{N}$ and $g_0 \in L_1$ such that $g_n \to_{n\in\mathbb{N}_2} g_0$ weakly. It suffices to prove that $g_0 = m_1$. By Lemma 6 and (13) we have

$$\int \Phi(|f-g_0|)\, d\mu \leqslant \varliminf_{n\in\mathbb{N}_2} \int \Phi(|f-g_n|)\, d\mu \leqslant \int \Phi(|f-m_1|)\, d\mu. \tag{19}$$

As $g_0 \in \mu_1(f \mid C)$ by (14), relation (19) implies $g_0 = m_1$ according to (ii).

Now we show that the condition $\varnothing \neq \mu_1(f \mid C) \subset L_{1+}$ used in Theorem 2, is fulfilled in important cases. The results of the following Lemma 3 were proven in [7].

3. LEMMA. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and $\varnothing \neq C \subset L_1$ be a $\| \|_1$-closed lattice. Then we have for all $f, g \in L_1$ that*

(i) *$\varnothing \neq \mu_1(f \mid C)$ has a minimum and a maximum, say $\underline{\mu}_1(f \mid C)$ and $\bar{\mu}_1(f \mid C)$,*

(ii) *$f \leqslant g \Rightarrow \underline{\mu}_1(f \mid C) \leqslant \underline{\mu}_1(g \mid C)$; $\bar{\mu}_1(f \mid C) \leqslant \bar{\mu}_1(g \mid C)$.*

Proof. (i) follows from Theorem 14 of [7], (ii) follows from Theorem 18 of [7].

4. PROPOSITION. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and $\varnothing \neq C \subset L_1$ be a $\| \|_1$-closed lattice with $aC + b \subset C$ for $a \geqslant 0$, $b \in \mathbb{R}$. Then $\varnothing \neq \mu_1(f \mid C) \subset L_{1+}$ for each $f \in L_{1+}$.*

Proof. Let $f \in L_s$ for some $s > 1$. According to Lemma 3(i) and (ii) it suffices to show that $\underline{\mu}_1(f \mid C) \in L_s$ and $\bar{\mu}_1(f \mid C) \in L_s$. For each $g \in L_1$ let $Tg := \bar{\mu}_1(g \mid C) \in L_1$. Then $T: L_1 \to L_1$ is a monotone operator according to Lemma 3(ii). Furthermore—using $aC + b \subset C$ for $a \geqslant 0$, $b \in \mathbb{R}$—it is easy to see that $T(ag + b) = a\, Tg + b$ for $a \geqslant 0$, $b \in \mathbb{R}$. These properties of T imply that

$$(T|f|)^s \leqslant T(|f|^s) \qquad \text{for} \quad f \in L_s \tag{20}$$

(compare for instance the proof of property (P.7) of [8]). As $|f|^s \in L_1$ we have $T(|f|^s) \in L_1$ whence (20) implies $\bar{\mu}_1(|f| \mid C) = T(|f|) \in L_s$. Since also $-C = \{-c : c \in C\}$ is a $\| \|_1$-closed lattice with $a(-C) + b \subset -C$ for $a \geqslant 0$, $b \in \mathbb{R}$, we also obtain $\bar{\mu}_1(|f| \mid -C) \in L_s$. As $\underline{\mu}_1(-g \mid C) = -\bar{\mu}_1(g \mid -C)$ we obtain using Lemma 3(ii) that

$$-\bar{\mu}_1(|f| \mid -C) = \underline{\mu}_1(-|f| \mid C) \leqslant \underline{\mu}_1(f \mid C) \leqslant \bar{\mu}_1(f \mid C) \leqslant \bar{\mu}_1(|f| \mid C).$$

As $\bar{\mu}_1(|f| \mid -C)$, $\bar{\mu}_1(|f| \mid C) \in L_s$ this implies $\underline{\mu}_1(f \mid C)$, $\bar{\mu}_1(f \mid C) \in L_s$.

If $\varnothing \neq C \subset L_1$ is a $\| \|_1$-closed lattice with $aC + b \subset C$ for $a \geqslant 0$, $b \in \mathbb{R}$, then according to Lemma 7.1 of [8] the set C is convex and there exists a σ-lattice $\mathscr{L} \subset \mathscr{A}$ such that $C = L_1(\mathscr{L})$. Hence C is the system of all equivalence classes of integrable functions which contain an $\mathscr{L}$-measurable function. These C's are exactly the systems considered in the theory of isotonic regression and approximation (see [2, 3, 4, 5, 8]). Proposition 4 shows that for these C's the assumption $\varnothing \neq \mu_1(f \mid C) \subset L_{1+}$ is fulfilled. Since $C \cap L_s = L_s(\mathscr{L})$ in this case the best $\| \|_s$-approximant $\mu_s(f \mid C \cap L_s)$ of f in $C \cap L_s$ is the element $\mu_s^{\mathscr{L}} f$ introduced in [8]. Using the properties of

$\mu_s^{\mathscr{L}} f$ proved in [8] one easily obtains from Theorem 2(iii) the following algebraic properties of Proposition 5 for $f \to m_1(f \mid C)$. (The assumption in [8] that μ was a probability measure instead of a finite measure does not matter.)

5. PROPOSITION. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and $\varnothing \neq C \subset L_1(\Omega, \mathscr{A}, \mu)$ be a $\| \|_1$-closed lattice with $aC + b \subset C$ for $a \geqslant 0$, $b \in \mathbb{R}$. Then for each $f \in L_{1+}$ the natural best $\| \|_1$-approximant of f in C, $m_1(f \mid C)$ exists and the map $f \to m_1(f \mid C)$ has the following properties.*

(i) $m_1(\cdot \mid C) \mid L_{1+}$ *is idempotent.*

(ii) $m_1(\cdot \mid C) \mid L_{1+}$ *is monotone.*

(iii) $m_1(af + b \mid C) = am_1(f \mid C) + b$ *for* $a \geqslant 0$, $b \in \mathbb{R}$, $f \in L_{1+}$.

(iv) *Let $\Phi: I \to \mathbb{R}$ be a non-decreasing continuous and convex function on a closed finite or infinite interval I. If $f(\Omega) \subset I$ and f, $\Phi \circ f \in L_{1+}$, then*

$$\Phi \circ (m_1(f \mid C)) \leqslant m_1(\Phi \circ f \mid C).$$

(v) $|m_1(f \mid C)| \leqslant \max(m_1(|f| \mid C), m_1(|f| \mid -C))$.

(vi) $m_1(\cdot \mid C)$ *maps L_r into L_r for each $r > 1$.*

If furthermore $-C \subset C$, then additionally

(vii) $m_1(g \cdot f \mid C) = gm_1(f \mid C)$ *for bounded functions $g \in C$ and $f \in L_{1+}$.*

(viii) $m_1(g + f \mid C) = g + m_1(f \mid C)$ *for bounded functions $g \in C$ and $f \in L_{1+}$.*

(ix) *The function Φ in* (iv) *need only be continuous and convex.*

Proof. According to Proposition 4 we have $\varnothing \neq \mu_1(f \mid C) \subset L_{1+}$ for all $f \in L_{1+}$. Hence according to Theorem 2, $m_1(f \mid C)$ exists and belongs to C, whence m_1 is idempotent, i.e., (i) holds. According to the remarks above there exists a σ-lattice $\mathscr{L}$ such that $C \cap L_s = L_s(\mathscr{L})$ and $\mu_s(f \mid C \cap L_s) = \mu_s^{\mathscr{L}} f$. According to Theorem 2(iii) there exists a sequence $s_n \downarrow 1$ such that $\mu_{s_n}^{\mathscr{L}}(f)$ converges μ-a.e. to $m_1(f \mid C)$. Hence (ii) follows from (2.8) of [8]. Property (iii) follows from (2.1) and (2.2) of [8]. Property (iv) follows from property (P.14) of [8]. Property (v) follows from (P.12) of [8] using that $-C = L_1(\bar{\mathscr{L}})$.

Ad property (vi): according to (v) we may assume $f \geqslant 0$. We remark that (vi) does not directly follow from (iv), applied to $\Phi(t) = t^r$, since (iv) is applicable only for f with $f^r \in L_{1+}$. We shall show that for $1 < s \leqslant 2$ and $s \leqslant r$

$$\int |\mu_s^{\mathscr{L}}(f)|^r \, d\mu \leqslant 2^{2-s} \int f^r \, dP. \tag{21}$$

From (21) we obtain (vi) using the Fatou lemma and $\mu_{s_n}(f \mid C) \to_{n\in\mathbb{N}} m_1(f \mid C)$ μ-a.e. for some appropriate sequence $s_n \downarrow 1$. Since according to (2.9) of [8] the operator $\mu_s^{\mathscr{L}}(f)$ is monotone continuous, we may assume that f is bounded. Using the convexity inequality (P.14) of [8] for the operator $\mu_s^{\mathscr{L}}(f)$ and the inequality $a^{s-1} \leqslant 2^{2-s}b^{s-1} - (b-a)^{\underline{s-1}}$ if $a, b \geqslant 0$, $1 < s \leqslant 2$ (see Lemma 7.2 of [8]), we obtain

$$\begin{aligned} \int |\mu_s^{\mathscr{L}}(f)|^r \, d\mu &= \int (|\mu_s^{\mathscr{L}}(f)|^{r/(s-1)})^{s-1} \, d\mu \\ &\leqslant \int |\mu_s^{\mathscr{L}}(f^{r/(s-1)})|^{s-1} \, d\mu \\ &\leqslant 2^{2-s} \int (f^{r/(s-1)})^{s-1} \, d\mu - \int (f^{r/(s-1)} - \mu_s^{\mathscr{L}}(f^{r/(s-1)}))^{\underline{s-1}} \, d\mu \\ &= 2^{2-s} \int f^r \, dP, \end{aligned}$$

where the last equality follows from (2.5) of [8].

Since $-C \subset C$ implies that $\mathscr{L}$ is a σ-field properties (vii), (viii) and (ix) follow from the corresponding properties of $\mu_s(f \mid C)$.

In the following we prove four lemmas which were needed in the proof of Theorem 2. A special case of Lemma 6 was proved in [6]. Lemma 7 may be of independent interest.

6. Lemma. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and $I \subset \mathbb{R}$ be a finite or infinite closed interval. Let $h_n \in L_1(\mu)$ with $h_n(\Omega) \subset I$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\Phi: I \to \mathbb{R}$ be a convex and continuous function. Then $h_n \to_{n\in\mathbb{N}} h_0$ weakly in L_1 implies $\int \Phi \circ h_0 \, d\mu \leqslant \underline{\lim}_{n\in\mathbb{N}} \int \Phi \circ h_n \, d\mu$.*

Proof. Since Φ is convex and $h_n \in L_1$ we have $\int \Phi \circ h_n \, d\mu > -\infty$ for $n \in \mathbb{N}_0$. W.l.g. $\alpha := \underline{\lim}_{n\in\mathbb{N}} \int \Phi \circ h_n \, d\mu < \infty$ and $\int \Phi \circ h_n \, d\mu \to_{n\in\mathbb{N}} \alpha$. Let $\alpha_k \downarrow \alpha$ and put $C_k := \{g \in L_1 \colon g(\Omega) \subset I \text{ and } \int \Phi(g) \, d\mu \leqslant \alpha_k\}$. It suffices to prove that C_k is weakly closed for all $k \in \mathbb{N}$. As C_k is convex, C_k is weakly closed if it is strongly closed. Let $g_n \in C_k$ with $g_n \to_{n\in\mathbb{N}} g_0$ strongly and w.l.g. $g_n \to_{n\in\mathbb{N}} g_0$ μ-a.e.; then $g_0(\Omega) \subset I$ and $\Phi \circ g_n \to_{n\in\mathbb{N}} \Phi \circ g_0$ μ-a.e.; if $\Phi \circ g_n$, $n \in \mathbb{N}$, is uniformly integrable from below, i.e.,

$$\sup_{n\in\mathbb{N}} \left| \int_{\{\Phi \circ g_n \leqslant \eta\}} \Phi \circ g_n \, d\mu \right| \to 0 \qquad \text{for} \quad \eta \to -\infty, \tag{22}$$

then $\int \Phi \circ g_0 \, d\mu \leqslant \underline{\lim}_{n\in\mathbb{N}} \int \Phi \circ g_n \, d\mu$ and hence $g_0 \in C_k$. Therefore it suffices to prove (22). As $\Phi(x) \geqslant ax+b$ for some $a, b \in \mathbb{R}$ we have for each $\eta < 0$

$$0 \geqslant \int_{\{\Phi \circ g_n \leqslant \eta\}} \Phi \circ g_n \, d\mu \geqslant a \int_{\{\Phi \circ g_n \leqslant \eta\}} g_n \, d\mu + b\mu\{\Phi \circ g_n \leqslant \eta\}. \tag{23}$$

As g_n, $n \in \mathbb{N}$, is uniformly integrable and Φ is convex, (23) implies (22).

7. LEMMA. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and $I \subset \mathbb{R}$ a finite or infinite closed interval. Let $h_n \in L_1(\Omega, \mathscr{A}, \mu)$ with $h_n(\Omega) \subset I$, $n \in \mathbb{N}_0$, and $\Phi: I \to \mathbb{R}$ be a strictly convex and continuous function. Then $h_n \to_{n \in \mathbb{N}} h_0$ weakly in L_1 and $\overline{\lim}_{n \in \mathbb{N}} \int \Phi(h_n) \, d\mu \leqslant \int \Phi(h_0) \, d\mu \in \mathbb{R}$ imply $h_n \to_{n \in \mathbb{N}} h_0$ strongly in L_1.*

Proof. As $h_n \to_{n \in \mathbb{N}} h_0$ weakly, h_n, $n \in \mathbb{N}$, is uniformly integrable. Hence it suffices to prove that h_n converges in measure to h_0. Since h_n converges to h_0 weakly, it suffices to prove that h_n, $n \in \mathbb{N}$, is Cauchy-convergent in measure. Assume indirectly that h_n, $n \in \mathbb{N}$, is not Cauchy-convergent. Then there exists $\varepsilon_0 > 0$ and a subsequence $g_k = h_{n_k}$ such that

$$\mu\{\omega : |g_k(\omega) - g_{k+1}(\omega)| \geqslant \varepsilon_0\} \geqslant 2\varepsilon_0 \qquad \text{for all} \quad k \in 2\mathbb{N}. \tag{24}$$

Since g_k, $k \in \mathbb{N}$, is uniformly integrable we have $\sup_{k \in \mathbb{N}} \int |g_k| \, d\mu < \infty$. Hence by the Markoff inequality there exists $a_0 > 0$ such that

$$\mu\{\omega : |g_k(\omega)| > a_0\} \leqslant \varepsilon_0/2 \qquad \text{for all} \quad k \in \mathbb{N}. \tag{25}$$

From (24) and (25) we obtain

$$\begin{aligned} \mu\{\omega : |g_k(\omega)|, |g_{k+1}(\omega)| \leqslant a_0, \\ |g_k(\omega) - g_{k+1}(\omega)| \geqslant \varepsilon_0\} \geqslant \varepsilon_0 \qquad \text{for all} \quad k \in 2\mathbb{N}. \end{aligned} \tag{26}$$

Since Φ is strictly convex and continuous and since I is a closed interval, we have

$$\begin{aligned} \gamma_0 := \inf \Big\{ \frac{1}{2} (\Phi(x) + \Phi(y)) - \Phi\left(\frac{x+y}{2}\right) : \\ x, y \in I, |x|, |y| \leqslant a_0, |x-y| \geqslant \varepsilon_0 \Big\} > 0. \end{aligned} \tag{27}$$

As $g_k \to_{k \in \mathbb{N}} h_0$ weakly in L_1, we obtain $\frac{1}{2}(g_k + g_{k+1}) \to_{k \in \mathbb{N}} h_0$ weakly in L_1 and hence by Lemma 6

$$\int \Phi(h_0) \, d\mu \leqslant \underline{\lim}_{k \in \mathbb{N}} \int \Phi(\tfrac{1}{2}(g_k + g_{k+1})) \, d\mu. \tag{28}$$

From Lemma 6 and our assumption we obtain $\int \Phi(g_k)\,d\mu \to_{k\in\mathbb{N}} \int \Phi(h_0)\,d\mu \in \mathbb{R}$. As Φ is convex, this implies

$$\int \Phi(\tfrac{1}{2}(g_k + g_{k+1}))\,d\mu$$

$$\leqslant \tfrac{1}{2}\int \Phi(g_k)\,d\mu + \tfrac{1}{2}\int \Phi(g_{k+1})\,d\mu \xrightarrow[k\in\mathbb{N}]{} \int \Phi(h_0)\,d\mu. \tag{29}$$

Now (28) and (29) imply

$$\tau_k := \int \left|\tfrac{1}{2}|\Phi(g_k) + \Phi(g_{k+1})| - \Phi(\tfrac{1}{2}(g_k + g_{k+1}))\right| d\mu \xrightarrow[k\in\mathbb{N}]{} 0.$$

On the other hand, (26) and (27) imply

$$\tau_k \geqslant \varepsilon_0 \gamma_0 \qquad \text{for all} \quad k \in 2\mathbb{N}$$

and we obtain a contradiction.

8. LEMMA. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and $h_n \in L_1(\mu)$, $n \in \mathbb{N} \cup \{0\}$. Then $h_n \to h_0$ weakly in L_1 and $\int |h_n|\,d\mu \to_{n\in\mathbb{N}} \int |h_0|\,d\mu$ imply $|h_n| \to |h_0|$ weakly in L_1.*

Proof. Let $A \in \mathscr{A}$ be given. Then $h_n 1_A \to_{n\in\mathbb{N}} h_0 1_A$ weakly and $h_n 1_{\bar{A}} \to_{n\in\mathbb{N}} h_0 1_{\bar{A}}$ weakly. Therefore

$$\int_A |h_0|\,d\mu \leqslant \varliminf_{n\in\mathbb{N}} \int_A |h_n|\,d\mu, \qquad \int_{\bar{A}} |h_0|\,d\mu \leqslant \varliminf_{n\in\mathbb{N}} \int_{\bar{A}} |h_n|\,d\mu \tag{30}$$

and we have

$$\varlimsup_{n\in\mathbb{N}} \int_A |h_n|\,d\mu = \varlimsup_{n\in\mathbb{N}} \left[\int |h_n|\,d\mu - \int_{\bar{A}} |h_n|\,d\mu\right]$$

$$= \int |h_0|\,d\mu - \varliminf_{n\in\mathbb{N}} \int_{\bar{A}} |h_n|\,d\mu \leqslant \int_A |h_0|\,d\mu.$$

Together with (30) this implies $\int_A |h_n|\,d\mu \to \int_A |h_0|\,d\mu$. As this holds for all $A \in \mathscr{A}$ we obtain $|h_n| \to |h_0|$ weakly.

9. LEMMA. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and $h_n \in L_1(\mu)$, $n \in \mathbb{N}_0$. Then $h_n \to_{n\in\mathbb{N}} h_0$ weakly in L_1 and $|h_n| \to_{n\in\mathbb{N}} |h_0|$ strongly in L_1 imply $h_n \to_{n\in\mathbb{N}} h_0$ strongly in L_1.*

Proof. Let $B_n = \{h_0 < 0 < h_n\}$, $C_n = \{h_n < 0 < h_0\}$ and $A_n = B_n \cup C_n$. Then

$$\int_{\bar{A}_n} |h_n - h_0| \, d\mu = \int_{\bar{A}_n} ||h_n| - |h_0|| \, d\mu \leqslant \int ||h_n| - |h_0|| \, d\mu \xrightarrow[n \in \mathbb{N}]{} 0.$$

Hence it suffices to prove that

$$\int_{A_n} |h_n - h_0| \, d\mu = \int_{A_n} (|h_n| + |h_0|) \, d\mu \xrightarrow[n \in \mathbb{N}]{} 0. \tag{31}$$

Our assumptions imply that

$$h_n^+ \xrightarrow[n \in \mathbb{N}]{} h_0^+ \text{ weakly}, \qquad h_n^- \xrightarrow[n \in \mathbb{N}]{} h_0^- \text{ weakly}. \tag{32}$$

We obtain from (32) that

$$\int_{B_n} |h_n| \, d\mu = \int_{\{h_0 < 0\}} h_n^+ \, d\mu \xrightarrow[n \in \mathbb{N}]{} \int_{\{h_0 < 0\}} h_0^+ \, d\mu = 0$$

and

$$\int_{C_n} |h_n| \, d\mu = \int_{\{h_0 > 0\}} h_n^- \, d\mu \xrightarrow[n \in \mathbb{N}]{} \int_{\{h_0 > 0\}} h_0^- \, d\mu = 0.$$

Hence

$$\int_{A_n} |h_n| \, d\mu \xrightarrow[n \in \mathbb{N}]{} 0. \tag{33}$$

As

$$\left| \int_{A_n} |h_n| \, d\mu - \int_{A_n} |h_0| \, d\mu \right| \leqslant \int ||h_n| - |h_0|| \, d\mu \xrightarrow[n \in \mathbb{N}]{} 0,$$

(33) implies

$$\int_{A_n} |h_0| \, d\mu \xrightarrow[n \in \mathbb{N}]{} 0. \tag{34}$$

Now (33) and (34) imply (31).

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